Notes on Geometric Measure Theory and Fractals

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16 Nov, 2014

Introduction

This notes is based on Prof. Boris Solomyak's lectures on the course: Geometric Measure Theory and Fractals as well as the reference book: Fractal sets in Probability and Analysis which written by Christopher J. Bishop and Yuval Peres.

Minkowski and Hausdorff Dimensions

Minkowski Dimension

Both Minkowski and Hausdorff dimensions measure *how efficiently a set K can be covered by balls*. **Minkowski dimension (Mdim)** requires that the covering be by balls all of the *same radius*. This is easy to compute while it lacks some certain desirable properties. In the definition of **Hausdorff dimension (Hdim)** we will allow coverings by balls of *difference radii*. This gives a better behaved notion of dimension, but is usually more difficult to compute.

Definition(Minkowski Dimension). For a *totally bounded set* K(can be finitely covered by balls with arbitrary diameter $\epsilon > 0$), let $N(K, \epsilon)$ denote the **minimal number of open balls of diameter ϵ needed to cover K.** We define the upper Mdim as

$$\overline{\dim}_M(K) = \limsup_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon},$$

and the lower Mdim as

$$\underline{\dim}_M(K) = \liminf_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon}.$$

If these two values agree, the common value is so-called Mdim or box counting dimension of K, and denoted by $\dim_M(K)$.

If we replace $N(K, \epsilon)$ by $N_D(K, \epsilon)$ where the covering sets of diameter $\leq \epsilon$ are not required to be balls, then we get the same values of upper Mdim and lower Mdim. For a bounded set K, $\overline{\dim}_M(K) = \overline{\dim}_M(\bar{K})$ and $\underline{\dim}_M(K) = \underline{\dim}_M(\bar{K})$

Examples:

(1) Mdim of a finite set is 0.
(2) Mdim of $[0, 1]$ is 1.
(3) For Cantor set, in the n-level, for $3^{-n} \leq \epsilon < 3^{-n+1}$, we have $N(E, \epsilon) \leq 2^n$, hence $\overline{\dim}_M(E) \leq \frac{\log 2}{\log 3}$. Conversely, any interval of length 3^{-n} can hit $E \cap I_n$ for at most two of the 2^n nth generation intervals I_n. Hence $N(E, 3^{-n}) \geq 2^{n-1}$, thus $\underline{\dim}_M(E) \geq \frac{\log 2}{\log 3}$
(4) If for every stage, we remove the middle $\alpha (0 < \alpha < 1)$, we get a Cantor set with Mdim $\log 2/(\log(2/(1-\alpha)))$.
(5) Countable sets can have positive dimension, and it's not true for $\dim_M(\cup E_n) = \sup_n \dim_M(E_n)$.

Hausdorff Dimension

The Hdim of K is defined to be

$$\dim_H(K) = \inf\{\alpha : \mathcal{H}^\alpha_\infty(K) = 0\}$$

where,

$$\mathcal{H}^\alpha_\infty(K) = \inf\{\sum_i |U_i|^\alpha : K \subset \cup_i U_i\},$$

$\{U_i\}$ is a countable cover of K by any sets and $|*|$ denotes the diameter, and $\mathcal{H}^\alpha_\infty(K)$ is called α-dimensional Hausdorff content. We will use $\dim(K)$ instead of $\dim_H(K)$ for simplicity later.

More generally, we have definition of Hausdorff Measure. We defined α-dimensional Hausdorff Measure as

$$\mathcal{H}^{\alpha}(K) = \lim_{\epsilon \to 0} \mathcal{H}^{\alpha}_{\epsilon}(K),$$

where

$$\mathcal{H}^{\alpha}_{\epsilon}(K) = \inf\{\sum_i |U_i|^{\alpha} : K \subset \cup_i U_i, |U_i| < \epsilon\},$$

Remarks:

1. α-dimensional Hausdorff measure is an outer measure.
2. α-dimensional Hausdorff measure in R^d is a Borel measure if $\alpha < d$.
3. If we only admit open sets in the covers of K, then $\mathcal{H}^{\alpha}_{\epsilon}(K)$ never changes. Using a balls cover will increase a 2^{α} factor.

Definition. A set is $\mathcal{H}^{\alpha}$-measurable if

$$\mathcal{H}^{\alpha}(A) = \mathcal{H}^{\alpha}(A \cap K) + \mathcal{H}^{\alpha}(A \cap K^c),$$

for any set A and any Borel set K.

Properties:

1. If $\mathcal{H}^{\alpha}(K) < \infty$ then $\mathcal{H}^{\beta}(K) = 0$ for any $\beta > \alpha$.
2. $\mathcal{H}^{\alpha}(K) = 0 \iff \mathcal{H}^{\alpha}_{\infty}(K) = 0$
3. $\dim_H(K) \leq \underline{\dim}_M(K)$

For Sierpinski Gasket, we have the Hausdorff dimension is $\frac{\log 3}{\log 2}$. Indeed, it's easy to compute the upper bound of Hdim via Mdim. We just need to verify the lower bound. To estimate the we should use **Mass Distribution Principle** for common cases.

Lemma(MDP) If E supports a strictly positive Borel measure μ, (i.e. $\exists \mu$ on R^d,s.t. $\mu(E) > 0$) which satisfies $\mu(B(x, r)) \leq Cr^{\alpha}$, for some constant $0 < C < \infty$ and for every ball $B(x, r)$, then $\mathcal{H}^{\alpha}(E) \geq \mathcal{H}^{\alpha}_{\infty}(E) \geq \mu(E)/C$. In particular, $\dim_H(E) \geq \alpha$.

Main Point: The upper bounds for Hausdorff dimension usually come from *finding explicit coverings* of the set, but lower bounds are proven by *constructing an appropriate measure supported on the set*.

Examples:

1. if $E \subset R^d, \mathcal{L}^d(E) > 0$, then $\dim_H(E) = d$.
2. Cantor set, use the natural probability measure on Cantor set. (unitary distribution) Suppose $3^{-n-1} < 2r \leq 3^{-n}$, then $B(x,r)$ hits at most one n-level interval, i.e. $\mu(B(x,r)) \leq 2^{-n}$, then we have $2^{-n} = (3^{-n})^\alpha$, i.e. $\alpha = \log 2/\log 3$.
3. The α Hausdorff measure of Cantor set is 1.

Sets defined by digit restrictions

In this section we will consider some more complicated sets (**which is s subset of** R) for which Mdim is easy to compute but the Hausdorff dimension is not so obvious.

First, we give a good example that Hausdorff measure is infinity, i.e. $K \subset \mathbb{R}^d$ is compact, $\dim_H K = \alpha$ with $\mathcal{H}^\alpha(K) = \infty$.

Example (Generalized Cantor Sets) at each stage the length of intervals are $\frac{1}{3} + r_n$, where r_n is a positive decreasing sequence going to 0. One can show that $\dim_H(K) = \alpha = \frac{\log 2}{\log 3}$. If $\sum r_n = +\infty$, then $\mathcal{H}^\alpha(K) = +\infty$.

Let $b \geq 2$ be an integer and consider b-adic expansions of real numbers. For each integer n $x \in [0,1]$, let $I_n x$ denote the unique interval of the form $\left[\frac{k-1}{b^n}, \frac{k}{b^n}\right)$ containing x. Such intervals are called b-adic intervals of generation n (dyadic if $b = 2$).

Frostman and Besicovitch had showed that we can restrict the infimum in the definition of Hausdorff measure to coverings of the set which only involve b-adic intervals and only change the value by a bounded factor.

We define the grid Hausdorff content by

$$\hat{\mathcal{H}}^{\alpha}_{\infty}(A) = \inf\{\sum_i |J_i|^{\alpha}, A \subset \cup_i J_i\},$$

and the grid Hausdorff measures by

$$\hat{\mathcal{H}}^{\alpha}_{\epsilon}(A) = \inf\{\sum_i |J_i|^{\alpha}, A \subset \cup_i J_i, |J_i| < \epsilon\},$$

where the infimums are over all coverings of $A \subset \mathbb{R}$ by collection J_i of b-adic intervals.

$$\hat{\mathcal{H}}^{\alpha}(A) = \lim_{\epsilon \to 0} = \hat{\mathcal{H}}^{\alpha}_{\epsilon}(A)$$

It's clear that $(b+1)\mathcal{H}^{\alpha}(A) \geq \hat{\mathcal{H}}^{\alpha}(A) \geq \mathcal{H}^{\alpha}(A)$, since every interval can be covered by $\leq (b+1)$ b-adic intervals. Hence, $\mathcal{H}^{\alpha}(A) = 0 \iff \hat{\mathcal{H}}^{\alpha}(A)$

The advantages of b-adic intervals are:

1. Any two b-adic intervals I, J either $I \cap J = 0$ or $I \subset J$ ($J \subset I$ resp.).
2. Any covering by b-adic intervals always contains a subcover by disjoint intervals (by taking the maximal intervals).
3. b-adic intervals can be given by the structure of a tree.

Definition For $S \subset \mathbb{N}$ the upper density of S is

$$\bar{d}(S) = \limsup_{n \to \infty} \frac{S \cap \{1, \ldots, n\}}{n},$$

similarly we can define the lower density. If the two values agree the limit exists and is called $d(S)$, the density of S.

The following example is very important based on the definition of density:

Example Suppose $S \subset \mathbb{N}$, and define $A_S = \{x = \sum_{k=1}^{\infty} x_k 2^{-k}\}$, where $x_k \in \{0, 1\}$ if $k \in S$, otherwise, $x_k = 0$. An easy observation is that the upper Minkowski dimension of A_s coincides $\bar{d}(S)$ (resp. Lower Mdim). In particular, if $S = \cup_{n=1}\infty[(2n)!, (2n+1)!]$, then $\bar{d}(S) = 1$, and $\underline{d}(S) = 0$.

Shift of finite type

The shifts of finite type are defined the restricting which digits can follow other digits. Let $A = (A_{ij})$ be a $b \times b$ matrix of 0s and 1s, and define

$$X_A = \left\{ \sum_{n=1}^{\infty} x_n b^{-n} : A_{x_n x_{n+1}} = 1 \text{for all } n \geq 1 \right\}.$$

Note that X_A is homogeneous since the related graph is strongly connected. Symbolically, X_A is the set of infinite paths in the graph. Then, the Hausdorff dimension of X_A is equal to its Minkowski dimension, more precisely, we have

$$\dim_M(X_A) = \frac{\log \rho(A)}{\log b},$$

where $\rho(A)$ is the spectral radius of the matrix A, and is equal to absolute value of the largest eigenvalue.

This type could be extended to plane. Let $Y_A = \{(x, y) : A_{y_n x_n} = 1 \text{for all } n\}$, where x_n, y_n are the b-ary expansions of x and y. For example, if $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $x_n = 0$ implies that y_n can be either 1 or 0, but if $x_n = 1$ then y_n must be 0. This matrix A gives a Sierpinski Gasket.

Billingsley's Lemma

Lemma (Billingsley's Lemma) Let $A \subset [0, 1]$ be Borel and let μ be a finite Borel measure on $[0, 1]$. Suppose $\mu(A) > 0$, if

$$\alpha \leq \liminf_{n \to \infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \leq \beta$$

for all $x \in A$, then

$$\alpha \leq \dim_H(A) \leq \beta.$$

An example of this lemma is to find the Hausdorff dimension of A_S. We claims that $\dim_H(A_S) = \underline{d}(S)$. To prove this, let μ be the probability measure on A_S that gives equal measure to the n-th generation covering intervals. Then for any $x \in A_S$,

$$\frac{\log \mu(I_n(x))}{\log |I_n(x)|} = \frac{\log 2^{-|S \cap \{1, \ldots, n\}|}}{\log 2^{-n}} = \underline{d}(S)$$

Dimension of measures

Definition(dimension of measures) If μ is a Borel measure on $\mathbb{R}^n$ we define

$$\dim(\mu) = \inf\{\dim_H(A) : \mu(A^c) = 0, \exists \varnothing \neq A \subset \mathbb{R}\}.$$

Alternatively,

$$\dim(\mu) = \inf\{\alpha : \mu \perp \mathcal{H}^\alpha\}.$$

The following lemma give the method to compute the dimension of measures:

Lemma Let b be a positive integer and given $x \in [0,1]$ let $I_n(x)$ denote the b-adic interval of the form $[\frac{j-1}{b^n}, \frac{j}{b^n})$ containing x. Let

$$\alpha_\mu = \text{esssup}\left\{\liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|}\right\},$$

where $\text{esssup}(A) = \min\{\alpha : \mu(A \cap (\alpha, \infty)) = 0\}$. Then $\dim(\mu) = \alpha_\mu$.

Sets defined by digit frequency

If we don't restrict the particular digits, but require that each digit occurs with a certain frequency. The resulting sets are dense in $[0,1]$, so we need only consider the Hausdorff dimension.

A example is following, Let

$$A_p = \left\{x = \sum_{n=1}^{\infty} x_n 2^{-n} : \lim_{j\to\infty} \frac{1}{j} \sum_{k=1}^{j} x_k = p\right\},$$

then A_p is a set of real numbers in [0,1] such that 1 occurs in the binary expansion about pth of the time. $A_{1/2}$ is a set of full Lebesgue measure in [0,1]. In general, we have

$$\dim_H(A_p) = \dim(\mu_p) = h_2(p) = -p\log_2 p - (1-p)\log_2(1-p).$$

The quantity h_2 is called the *entropy* of p and is strictly less that 1 except for $p = 1/2$. It represents the uncertainty associated to the probability (if $p = 0, 1$ then entropy is 0, and it is maximal when $p = 1/2$)

Slices

If we have a higher dimensional set A which has dimension α, then we want to know the dimension of $A \cap L$, where L is random cut (line, plane etc.) Now we consider the 2 dimension case, set $A_x = \{y : (x, y) \in A\}$. We have following theorem:

Theorem(Marstrand Slicing Theorem) Let $A \subset \mathbb{R}^2$ and suppose that $\dim_H(A) \geq 1$. Then

$$\dim(A_x) \leq \dim_H(A) - 1,$$

for Lebesgue almost every x.

Remarks. If $\dim(A) < 1$ then the slice A_x is empty for almost every x (it's empty except for a set of dimension $\dim(A)$). Also, it's possible that $\dim(A_x) = \dim(A)$ for some value of x. To prove this theorem, we need to use the integral property, i.e. $\int_{\mathbb{R}} \mathcal{H}^{\alpha-1}(A_x)dx \leq \mathcal{H}^{\alpha}(A)$

Example. Let $A = C \times C$ be the product of the middle thirds Cantor set with itself, then $\dim(A) = \log_3 4$. The vertical slices of A are empty almost surely, so $\dim(A_x) = 0 < (\log_3 4 - 1)$ almost surely.

Another example is the Sierpinski Gasket G, we claim that almost surely every slice G_x of G has dimension $1/2 < \log_2 3 - 1$. Note that $G = \{(x, y) : x_n = 1, y_n = 0; x_n = 0, y_n = \{0, 1\}\}$, hence $G_x = A_{S(x)}$, where $S(x) = \{n : x_n = 0\}$. From Law of Large Numbers, digits 0 and 1 occur with equal frequency in its binary expansion, hence $\dim(G_x) = 1/2$.

Self-similarity

As we known in last chapter, the Minkowski dimension may not exist, and even it's exist. There is no guarantee that Minkowski dimension is equal to its Hausdorff dimension. In this chapter, we will consider various conditions on a compact set K which ensure that the Mdim exists and equals the Hdim.

Iterated Functions System

Definition: A contracting self map (or **contraction**) of a metric space (X, d) is a function $f : X \to X$ if $d(f(x), f(y)) \leq rd(x, y)$, for some $0 < r < 1$. A family of contractions $\{f_i\}_{i=1}^m$, $m \geq 2$ is called **Iterated Functions System (IFS)**. A nonempty compact set $K \subset X$ is called an **attractor** for IFS if $K = \cup_{i=1}^m f_i K$.

Examples: The Cantor set C is an attractor for ISF:$\{\frac{x}{3}, \frac{x+2}{3}\}$. The Sierpinski gasket(right angle triangle) which is an attractor of ISF: $\{(x, y)/2, (x + 1, y)/2, (x, y + 1)/2\}$.

The following theorem is well-known which proved by Hutchinson (1981).

Theorem. Let $\{f_i\}_{i=1}^m$ be IFS on a complete metric space (X, d), then

1. There exists a unique nonempty compact set $K \subset X$ is the attractor.
2. For any probability vector $\vec{p} = (p_1, p_2, \ldots, p_m)$, there exists a unique probability measure $\mu = \mu_p$ (called the stationary measure) on the attractor K such that

$$\mu = \sum_{i=1}^m p_i \mu f_i^{-1}$$

   If $p_i > 0$ for all $i \leq m$, then $supp(\mu) = K$.

Infinite Product Space

Definition: Infinite Product Space $\Omega_m = \{1, \ldots, m\}^{\mathbb{N}}$ is the collection of element ω, such that ω is an infinite sequence of $\{1, \ldots, m\}$, i.e. $\omega = \omega_1\omega_2\ldots$, where $\omega_i \in \{1, \ldots, m\}$. The metric defined on Ω_m is $d(\omega, \tau) = e^{-|\omega\wedge\tau|}$, where $|\omega \wedge \tau|$ is the length of the longest initial segment on which the two sequences agree. For example, if $\omega = 123423431\ldots$ and $\tau = 123523431\ldots$, then $|\omega \wedge \tau| = |123| = 3$.

Then we can define a map (called natural projection map): $\Phi : \{1, \ldots, m\}^{\mathbb{N}} \to K$, where $\Phi(\omega) = \cap_{n=1}^\infty K_{\omega(n)}$, $\omega(n)$ is the first n-th sequence, and $K(\omega(n)) = f_{\omega_1} \circ f_{\omega_2} \circ f_{\omega_n}(K)$. The mapping is **(Holder) continuous and onto**.

In a word, Ω_m provides addresses for the points in the attractor. Since, the map is **not always injective**, so sometimes such address is not unique. For example, any point in cantor set can bee uniquely represented by $\omega \in \{0,1\}^{\mathbb{N}}$; while for Sierpinski gasket, the center point of the bottom edge has two representation on $\{0,1,2\}^{\mathbb{N}}$, say 12^{∞} and 21^{∞}.

Proposition The natural projection map is injective if and only if

$$f_i(K) \cap f_j(K) \neq \varnothing, \forall i \neq j$$

Remark. For the attractor of IFS $\{\lambda x, \lambda x+1\}$, if $\lambda < \frac{-1+\sqrt{5}}{2}$, then there are existing points inside the attractor which have unique addresses. If $\lambda > \frac{-1+\sqrt{5}}{2}$, then except end points (fix points of IFS), all other points have uncountable many addresses.

Given a probability vector $\vec{p} = (p_1, p_2, \ldots, p_m)$ define a measure (called product or Bernoulli measure) on Ω_m, $\nu_p = (p_1, \cdots, p_m)^{\mathbb{N}}$.

Lemma. The stationary measure for IFS coincides the push-backward of product measure. Namely, $\mu_p = \nu_p \Phi^{-1}$.

Self-similar sets and measures

Definition. A mapping $f : X \to X$ is a **similitude** if $\exists r > 0$ such that $\forall x, y \in X$, $d(f(x), y(x)) = rd(x,y)$. when $r < 1$, the ratio r is called a contraction ratio. If IFS are all similitudes, then the attractor K is called a *self-similar set*.

Remarks. Let IFS $\{f_i\}_{i=1}^m$ be similitudes, assuming the set $\{f_i(K)\}_{i=1}^m$ are disjoint and $0 < \mathcal{H}^{\alpha}(K) < \infty$, it follows that

$$1 = \sum_{i=1}^{m} r_i^{\alpha}.$$

Definition. In general, For IFS the unique $\alpha > 0$ satisfying above relation (no need the other two assumptions) is called the **similarity dimension**.

Definition. An IFS on metric space X satisfies **open set condition(OSC)** if there is a bounded open nonempty set $V \subset X$ such that $f_i(V) \subset V$ for $1 \leq i \leq m$ and $f_i(V) \cap f_j(V) = \varnothing$ for $i \neq j$.

Moran and Hutchinson showed that OSC on $\mathbb{R}^d$ implies the Hausdorff dimension of the attractor is equal to similarity dimension as well as Minkowski dimension.

Theorem(Moran, Hutchinson) Let $\{f_i\}_{i=1}^m$ be contracting similitudes of Euclidean space $\mathbb{R}^d$ and let K be the corresponding attractor. Let α be the similarity dimension determined by IFS. If IFS satisfy the open set condition then $0 < \mathcal{H}^\alpha(K) < \infty$ and $\dim_H(K) = \alpha = \dim_M K$.

Definition. A set of finite strings $\pi \subset \cup_{n=1}^{\infty}\{1, 2, \dots, m\}^n$ is a **cut-set** if every infinite sequence in Ω_m has a prefix in π. The set of strings π is a minimal cut-set if no element of π is a prefix of another.

Remark. Graphically, the cut-set is the set cut-off all flow via tree. If before any cut-point of the cut-set, the flow is nonzero.

Lemma Let π be a minimal cut-set and let $p = (p_1, \dots, p_m)$ be a probability vector. Then $\sum_{\sigma\in\pi} p_\sigma = 1$. what's more, if μ is a measure satisfying $\mu = \sum_{i=1}^m p_i \mu f_i^{-1}$, then $\mu = \sum_{\sigma\in\pi} p_\sigma \mu f_\sigma^{-1}$

A more strict condition is **Strong separation condition(SSC)** that $f_i(K) \cap f_j(K) = \varnothing (i \neq j)$. It's clear that SSC implies OSC. On the contrary, Schief shows that if K is an attractor of an IFS of similitudes with Similarity dimension(Sdim) $\alpha > 0$ and $\mathcal{H}^\alpha(K) > 0$, then OSC holds. i.e. we usually can not drop OSC.

Homogeneous sets

In this section, we will introduce a new kind of set which, in some sense, look the same at all scales.

Definition. A compact set K is b-homogeneous if $T_b K = K$, where for the integer $b > 0$ defined the b-to-1 map T_b mapping [0,1] to itself by $T_b(x) = bx \mod 1$

Lemma(Furstenberg's Lemma) If $K \subset [0,1]$ is compact and $T_b(K) = K$, then $\dim_H(K) = \dim_M(K)$.

Self-affine sets

We modify the previous example by taking a non-square matrix. Suppose $m < n$ and suppose A is an $m \times n$ matrix of 0s and 1s with rows and columns labelled from 0 to m-1 (resp. n-1). Let $Y_A = \{(x,y) : A_{y_k x_k} = 1 \text{for all } k\}$, where x_k is the n-ary expansion of x and y_k is the m-ary expansion of y.

A famous set McMullen set is defined by a matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. We will discuss this kind of sets latter. For McMullen set, the Hausdorff dimension is $\log_2(2^{\log_3 2}+1) \approx 1.34968$ and Minkowski dimension is $1 + \log_3 \frac{3}{2} \approx 1.36907$

Frostman's Theory

Frostman's Lemma

In the previous methods, we use proper measure to compute the Hausdorff dimension, including using the Mass Distribution Principle and Billingsley's Lemma. Frostman's theory is somehow that is not too concentrated and use this powerful idea to derive a variety of results including the products, projections and slices.

Before we study the new theory, we need the definition of gauge functions (dimension functions), which are *a generalization of the simple diameter to the dimension power law* used in the construction of s-dimensional Hausdorff measure. The gauge function

$\varphi : [0, \infty] \to [0, \infty]$ should be monotonically increasing for $t \geq 0$, strictly positive for $t > 0$, and continuous on the right for all $t \geq 0$.

Then the Hausdorff dimension and measure corresponding to φ:

$$\mathcal{H}^{\varphi}(K) = \lim_{\epsilon \to 0} \mathcal{H}^{\varphi}_{\epsilon}(K),$$

$$\mathcal{H}^{\varphi}_{\epsilon}(K) = \inf \left\{ \sum_{i=1}^{\infty} \varphi\left(\text{diam}(C_i)\right) \middle| \text{diam}(C_i) \leq \epsilon, \bigcup_{i=1}^{\infty} C_i \supseteq K \right\}.$$

Lemma(Frostman's Lemma). Let φ be a gauge function. Let $K \subset \mathbb{R}^d$ be a compact set with positive Hausdorff content, $\mathcal{H}^{\varphi}_{\infty}(K) > 0$. Then there exists a positive Borel measure μ on K such that

$$\mu(B) \leq C_d \varphi(|B|),$$

for all balls B and $\mu(K) \geq \mathcal{H}^{\varphi}_{\infty}(K)$. Here C_d is a positive constant depending on d.

Dimension of product sets

Theorem. If $A \subset \mathbb{R}^d$,$B \subset \mathbb{R}^n$ are compact, then

$$\dim(A) + \dim(B) \leq \dim(A \times B) \leq \dim(A) + \overline{\dim}_M(B).$$

Note that for the upper bound on $\dim(A \times B)$, we don't need the compactness of either A or B. The left hand inequality of above theorem is Marstrand's Product theorem and the right hand side has a more good result, namely $\dim_p(B)$ (package dimension) instead of upper Minkowski dimension, which is due to Tricot, refining the early result of Bosicovitch and Moran.

Example. Let $S \subset \mathbb{N}$, then $\bar{\dim}_M(A_S) = \bar{\dim}_M(A_{S^c}) = 1$ and $\dim(A_S) = \dim(A_{S^c}) = 0$. Hence,

$$0 \leq \dim(A_S \times A_{S^c}) \leq 1.$$

We claim that $\dim(A_S \times A_{S^c}) = 1$, since $(x, y) \mapsto x + y$ is a Lipschitz function, hence $\dim(A_S \times A_{S^c}) \geq 1$

Capacity and dimension

Motivated by Frostman's Lemma, we make following definition.

Definition(energy). For a Borel measure μ on $\mathbb{R}^d$ and $\alpha > 0$ we define the α-dimensional energy of μ to be

$$\mathcal{E}_\alpha(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha},$$

and for a set $K \subset \mathbb{R}^d$ we define the α-dimensional capacity of K to be

$$\left[\inf_\mu \mathcal{E}_\alpha(\mu)\right]^{-1},$$

where the infimum is *over all Borel probability measures* supported on K. If $\mathcal{E}_\alpha(\mu) = \infty$ for all such μ, then we say $\mathrm{Cap}_\alpha(K) = 0$.

The following theorem shows that $\dim(K) = \inf\{\alpha : \mathrm{Cap}_\alpha(K) = 0\}$.

Theorem(Frostman 1935). Suppose $K \subset \mathbb{R}^d$ is compact. If $\mathcal{H}^\alpha(K) > 0$ then $\mathrm{Cap}_\beta(K) > 0$ for all $\beta < \alpha$. If $\mathrm{Cap}_\alpha(K) > 0$, then $\mathcal{H}^\alpha(K) = \infty$.

Marstrand Projection Theorem

Let $K \subset \mathbb{R}^2$ be a compact set and let Π_θ be the projection of $\mathbb{R}^2$ onto the line through the origin in the direction $(\cos\theta, \sin\theta)$, for $\theta \in [0,\pi)$. Since the projection is Lipschitz and $\prod_\theta K \subset \mathbb{R}$, $\dim(\prod_\theta K) \leq \min\{\dim(K), 1\}$. The Marstrand theorem states that for almost all directions the equality holds.

Theorem. If $\mathrm{Cap}_\alpha(K) > 0$ for some $0 < \alpha < 1$, then for almost every θ, the capacity $\mathrm{Cap}_\alpha(\prod_\theta K) > 0$.

This has the following corollary.

Corollary. If $\dim(K) \leq 1$, then $\dim(\prod_\theta K) = \dim(K)$ for almost every θ.

If K has dimension 1, then Marstrand's theorem says $\dim(\prod_\theta K) = 1$ for almost every θ. However, even if $\mathcal{H}^1(K)$ is positive this does not guarantee that $\mathcal{L}_1(\prod_\theta K)$ is positive. For example, the standard 1/4-Cantor set in the plane has projections of zero length in almost all directions. But we have following result instead.

Theorem. If $\mathrm{Cap}_1(K) > 0$, then $\mathcal{L}_1(\prod_\theta K) > 0$ for almost every θ.

In higher dimension, i.e. $K \subset \mathbb{R}^n$, define **Grassmannian** $G(n,m)$ which is the set of all m-dimensional subspace of $\mathbb{R}^n$. For $V \in G(n,m)$, define P_V to be orthogonal projection: $\mathbb{R}^n \to V$. The following theorem is the generalization of Marstrand Projection Theorem.

Theorem. Suppose $K \subset \mathbb{R}^n$, then $\dim(P_V K) = \min\{\dim K, m\}$ for a.e. $V \in G(n,m)$. What's more, if $\dim(K) > m$, then $\mathcal{L}_m(P_V K) > 0$ for a.e. $V \in G(n,m)$.

As for product of the generalized Cantor sets, Peres and Schmertain (2008) had proved that

$$\dim(P_\theta(C_\alpha \times C_\beta)) = \min\left\{\frac{\log 2}{\log(1/\alpha)} + \frac{\log 2}{\log(1/\beta)}, 1\right\},$$

if $\log\alpha/\log\beta \notin \mathbb{Q}$, then the above equation holds for $\theta \notin \{0, \pi/2\}$.

Self-affine sets

We have know some self-affine sets before, like McMullen set. For this kind of sets, the two dimensions are usually different. These sets are invariant under affine maps of the form $(x,y) \to (nx, my)$, sometimes with translations.

Construction and Minkowski dimension

Definition(Self-affine sets). The self affine set is attractor of IFS $\{T_i x + b_i\}_{i=1}^m$, where T_i : are linear contractions. (i.e. for any norm $||T_j|| < 1$, namely absolute of the maximal eigenvalue is less than 1.)

Remarks. Self-similar set is defined to be a self-affine set with T_j are r_j orthogonal, say $T_j = r_j O_j$, $r_j \in (0,1)$. Commonly, self-similar set has the same retraction on different directions. There is no equation $\sum r_j^\alpha = 1$ for self-affine set satisfying OSC.

Then, we generalize definition of McMullen set. Take $D \subset \{0,1,\dots,n-1\} \times \{0,1,\dots,m-1\}$, assume $n \geq m$, $T_1 = \cdots = T_m = \begin{bmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{bmatrix}$. Let

$$K(D) = \left\{ \sum_{k=1}^{\infty} (a_k n^{-k}, b_k m^{-k}) : (a_k, b_k) \in D \right\}.$$

Then, $K(D)$ is the attractor of IFS

$$\left\{ \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_k \\ b_k \end{pmatrix} \right\}$$

For example, for McMullen set, $n = 3, m = 2$ and $D = \{(0,0), (1,1), (2,0)\}$. The following theorem give the Minkowski of these kinds of sets:

Theorem. Suppose every row contains a chosen rectangle, $n > m$, then

$$\dim_M(K(D)) = 1 + \log_n \frac{\#D}{m}.$$

In general, if π denote the projection onto the second coordinate, so $\#(\pi(D))$ is the number of occupied rows (i.e. not empty), we get

$$\dim_M(K(D)) = \log_m \#(\pi(D)) + \log_n \frac{\#D}{\#(\pi(D))}$$

Then, the following theorem gives the result of Hausdorff dimension:

Theorem. Suppose every row contains a chosen rectangle, $n > m$, then

$$\dim(K(D)) = \log_m \left(\sum_{j=1}^{m} r(j)^{\log_n m} \right),$$

where $r(j)$ is the number of rectangles of the pattern lying in the j-th row.

The above two theorems show that

$$\dim(K(D)) < \dim_M(K(D)) \iff r(j) \text{ is not a constant.}$$

Heuristically, since we could use different sizes of covering sets, and the selected rectangles are not squares.

Hausdorff dimension of self-affine sets

To prove the theorem for Hausdorff dimension, we need to use approximate squares to calculate. Let $m < n$, and $\alpha = \frac{\log m}{\log n} < 1$.

Definition. Suppose $(x, y) \in [0, 1)^2$ have base n and base m expansions $\{x_k\}, \{y_k\}$, respectively. The approximate square of generation k at (x, y), $Q_k(x, y)$, is defined to be the closure of the set of points $(x', y') \in [0, 1)^2$ such that the first $\lfloor \alpha k \rfloor$ digits in the base n expansions of x and x' coincide, and first k digits in the base m expansions of y and y' coincide.

The rectangle $Q_k(x, y)$ as an approximate square of generation k since its width $n^{-\lfloor \alpha k \rfloor}$ and height m^{-k} satisfy:

$$m^{-k} \leq n^{-\lfloor \alpha k \rfloor} \leq nm^{-k}$$

and hence

$$m^{-k} \leq \text{diam}(Q_k(\omega)) \leq (n + 1)m^{-k}$$

The proof of the theorem could be found in Bishop and Peres's book (4.2). Then we consider the Hausdorff measure.

Definition. Define the projection π as $\pi(i,j) = j$. A digit set $D \subset \{0,1,\ldots,n-1\} \times \{0,1,\ldots,m-1\}$ has uniform horizontal fibres if all nonempty rows in D have the same cardinality. Otherwise, D has nonuniform horizontal fibres.

Then we have a theorem on Hausdorff measure of $K(D)$.

Theorem. For any gauge function φ, the Hausdorff measure $\mathcal{H}^{\varphi}(K(D))$ is either 0 or ∞, provided D has nonuniform horizontal fibres. In particular, the Hausdorff measure $\mathcal{H}^{\alpha}(K(D))$ is either 0 or ∞ with the same condition.

An elective topic is about singularity dimension of general self-affine sets which inspired by K. Falconer. To end this topic, we give an open problem on self-affine sets.

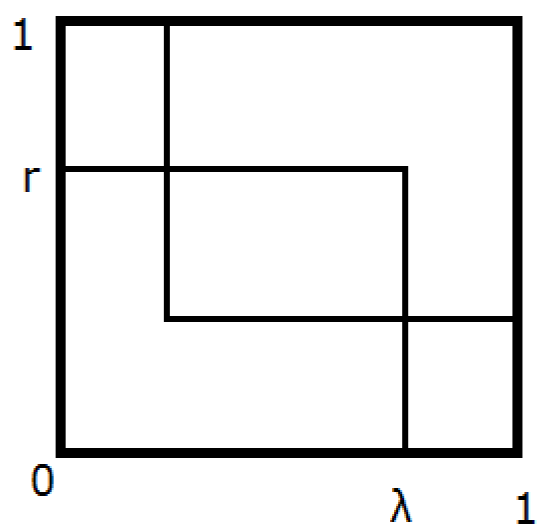

Figure 1: The self-affine set

In this problem, $T_1 = T_2$, with $1 > \lambda > r > 1/2$, suppose the limit set of this IFS is K. What is the condition on λ, r, such that K is continuous.

Note that if $\lambda + r = 1$, then the contractor is a continuous strictly increasing function with Hausdorff dimension 1.

Besicovitch-Kakeya Set

Definition.(Besicovitch Set) A Besicovitch set[1] $K \in \mathbb{R}^d$ is closed set of zero d-measure that contains a line segment in every direction.

A Kakeya set [2] which comes from Kakeya needle problems means a set where a line segment can **continuously** moved so as to return to its original position in the opposite direction. Here is an example for two dimension. It's easy to see a zero measurable Kakeya set is also a Besicovitch set.

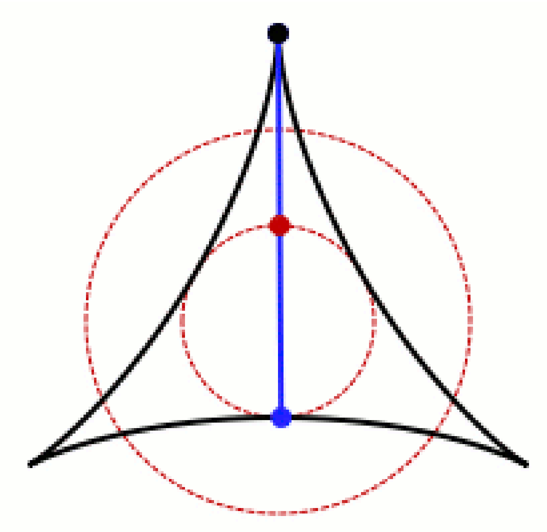

Figure 2: Kakeya needle

The Kakeya needle problems asks for the plane figure of least area in which a line segment of width 1 can be freely rotated (where translation of the segment is also allowed). Surprisingly, there is no minimum area (Besicovitch 1928).

When the figure is restricted to be **convex**, the smallest region is an **equilateral triangle of unit height**. Wells (1991) states that Kakeya discovered this, while Falconer (1990) attributes it to Pal.

[1]This definition is coming from Besicovitch Problem: Suppose $f(x,y)$ is Riemann integrable in the plane $\mathbb{R}^2$, is possible to find orthogonal axis's, such that $\int f(x,y)dy$ exists as Riemann integral function and $x \mapsto \int f(x,y)dy$ is Riemann Integral.

[2]Indeed, A Besicovitch set is also called Kakeya set, but it means another set in this note for simplicity.

Existence

Theorem.(Besicovitch 1928) There is a compact set $K \subset \mathbb{R}^2$ that has zero area and contains a unit line segment in every direction.

We omit the proof, however, we will give an example of Besicovitch set.

Theorem.(Besicovitch) If $F \subset \mathbb{R}^2$, $0 < \mathcal{H}^1(F) < \infty$, and there exists $\theta_1 \neq \theta_2$, such that $\mathcal{L}^1(P_{\theta_i}(F)) = 0$, then for almost all projections $\mathcal{L}^1(P_{\theta_i}(F)) = 0$.

Example. Consider $C_{1/4} := \{\sum_{n=1}^{\infty} a_n 4^{-n} : a_n \in \{0, 3\}\}$ is a Cantor set with $1/4$ in two sides. Consider all points in $C_{1/4} \times \{0\}$ with all points in $\frac{1}{2}C_{1/4} \times \{1\}$ by line segment. Here is a figure for first step of Cantor Set. We claim that the set K of those segments is a Besicovitch set.

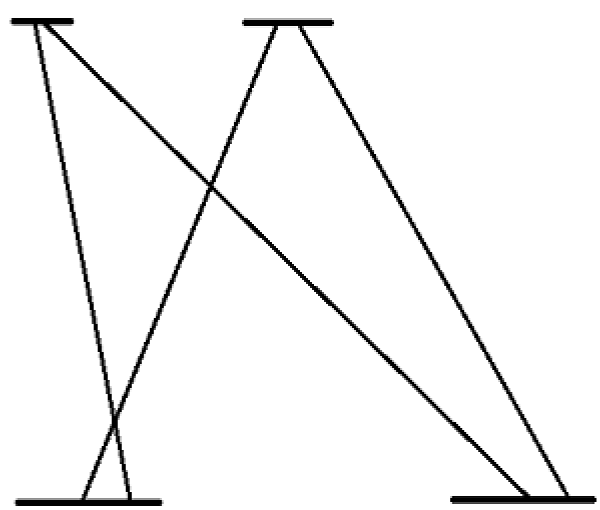

Figure 3: $C_{1/4}$ Besicovitch set

Remark. The set of 1/slope of K:

$$\{x - y : x \in C_{1/4}, y \in \frac{1}{2}C_{1/4}\} = C_{1/4} - \frac{1}{2}C_{1/4} = C_{1/4} + \frac{1}{2}C_{1/4} - \frac{1}{2} = \frac{3}{2}[0,1] - \frac{1}{2} = [-\frac{1}{2}, 1]$$

As for the area 0, we just need to show the horizontal line $K_t = \{t \sum a_n 4^{-n} + \frac{1-t}{2} \sum b_n 4^{-n}\}$ $(0 < t < 1, a_n, b_n = 0$ or $3)$ with $\mathcal{L}^1(K_t) = 0$ a.e. Then by Besicovitch theorem (the second one), we can get that K_t is zero length.

Dimension

As for the Hausdorff dimension, we have following theorem:

Theorem.(Daniel) If $K \subset \mathbb{R}^2$ is a compact set containing a segment in a compact set E of direction, then $\dim(K) \geq 1 + \dim(E)$. In particular, if K contains a segment in every dimension, then $\dim(K) = 2$.

Sums of Cantor sets

Introduction and First results

We consider arithmetic sums (or sumsets): $A + B = \{a + b : a \in A, b \in B\}$. Similarly, one can consider differences $A - B$, which is, of course, equivalent, since $A - B = A + (-B)$. Such sets come up in different areas of mathematics. For instance, if A and B are the supports of two measures μ, ν, respectively, then $A + B$ is the support of the convolution $\mu * \nu$.

The following theorem is a well-known result:

Steinhaus Theorem If $A, B \subset \mathbb{R}$ are Lebesgue measurable of positive measure, then $A \pm B$ has nonempty interior (contains an interval); in fact, $A - B$ contains a neighbourhood of the origin.

In particular, there is a famous Palis-Takens problem which studies is it true that $\mathcal{L}^1(C_\alpha + C_\beta) > 0$ implies that $C_\alpha + C_\beta$ contains an interval. There is a sufficient condition for the sum (or difference) of Cantor sets to contain an interval, which comes from the **Newhouse Gap Lemma** (M.Hall 1947).

Definition.(Gaps,Thickness) Let K be any Cantor set in $\mathbb{R}$. K is compact, perfect (no isolated point) and $K^0 = \varnothing$. Then the gaps of K are components of $\mathbb{R} \setminus K$. Given a bounded gap u, define its left and right bridge L_u(resp. R_u) be the largest left(resp. right) adjacent to u that does not intersect a gap of length $\geq |u|$. Define $\tau_u(K)$ as

$$\tau_u(K) = \min\left\{\frac{|L_u|}{|u|}, \frac{|R_u|}{|u|}\right\}$$

The thickness of K is

$$\tau(K) = \inf\{\tau_u(K) : u \text{ is the bounded gap of } K\}$$

An easy example for thickness of simple Cantor set is $\tau(C_\alpha) = \frac{\alpha}{1-2\alpha}$. What's more, we have following proposition.

Proposition. Given any Cantor set,

$$\dim(K) \geq \frac{\log 2}{\log(2 + \tau(K)^{-1})}$$

In particular, if $\tau(K) \to \infty$, then $\dim(K) \to 1$

Note that if dimension of K is closed to 1, we can not guarantee $\tau(K)$ is big enough. Indeed, consider a Cantor set with left interval has length α, right interval has length β for the first step. If $\alpha + \beta \approx 1$, but less than 1, then $\tau(K)$ is small. Now we are going to introduce Gap lemma. K_1, K_2 are sayed to linked if $K_1 \nsubseteq$ gaps of K_2, $K_2 \nsubseteq$ gaps of K_1.

Lemma.(Gap Lemma) Given Cantor sets K_1, K_2 are linked, and $\tau(K_1)\tau(K_2) > 1$, then $K_1 \cap K_2 \neq \varnothing$.

A direct corollary is that $K_1 \pm K_2$ contains intervals. we could from this corollary to get above assertions for simple Cantor sets. If $\alpha \geq \frac{1}{3}$, then $C_\alpha + C_\alpha = [0, 2]$. More generally, If $\tau(C_\alpha)\tau(C_\beta) \geq 1$, then $C_\alpha + C_\beta = [0, 2]$. From this we have a good figure for sum of simple Cantor sets.

In area A, $C_\alpha + C_\beta$ contains intervals, the boundary of A is $\tau(C_\alpha)\tau(C_\beta) = 1$, while in area B, $\dim(C_\alpha + C_\beta) < 1$, the boundary of B is $\frac{\log 2}{\log \alpha} + \frac{\log 2}{\log \beta} = -1$. This leaves a ``mysterious region'' between the two curves, where we don't know exactly what happens with Lebesgue measure and interior (where the Palis-Takens problem is non-trivial).

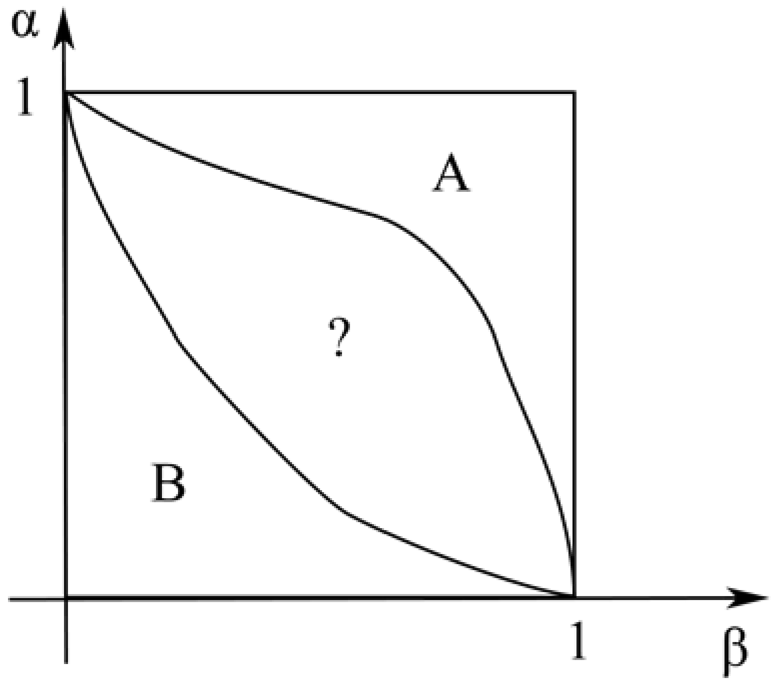

Figure 4: Sum of simple Cantor sets

Peres-Shmerkin theorem

The theorem of Y. Peres and P. Shmerkin about sums of Cantor sets, (see Y. Peres and P. Shmerkin, Resonance between Cantor sets, 2009) Although their result is a bit more general, we will show the corollary for sum and difference of Cantor sets as follows.

Theorem(Peres-Shmerkin). If $\log \alpha / \log \beta \notin \mathbb{Q}$, then

$$\dim_H(C_\alpha + C_\beta) = \min\{1, s_\alpha + s_\beta\},$$

where s_α, s_β are similarity dimensions for C_α and C_β respectively.
If $\log \alpha / \log \beta \in \mathbb{Q}$, $s_\alpha + s_\beta \leq 1$, then

$$\dim_H(C_\alpha + C_\beta) < s_\alpha + s_\beta.$$

Julia Sets and the Mandelbrot Set

Julia Sets

We consider $f : \mathbb{C} \to \mathbb{C}$ of the form $f(z) = z^2 + c$ for some c, and study the behaviour of iterates, i.e. the sequence $z, f(z), f^2(z), f3(z), \ldots$ and in particular $f^n(z)$ as $n \to \infty$ for various initial $z \in C$.

Definition.(Julia sets) For $f(z) = z^2 + c$ we define

$$F = \{z : |f^n(z)| \nrightarrow \infty\} = \{z : f^n(z) \text{ is a bounded sequence}\}$$

to be filled-in Julia set of f. We define the Julia set J of f to be the boundary of the filled-in Julia set, i.e.

$$J = \bar{F} \setminus \text{int}(F)$$

Properties of Julia Sets

1. J is non-empty closed and bounded, i.e. it is compact.
2. $J = f(J), J = f^{-1}(J)$
3. J contains no isolated points.
4. J is uncountable.

Mandelbrot Sets

We define the Mandelbrot set $M := \{c \in \mathbb{C} : \text{ the Julia set of } f(z) = z^2 + c \text{ is connected}\}$. Alternatively, $M := \{c \in \mathbb{C} : J_c \text{ is connected}\}$.

Fundamental Theorem. $c \in M \Leftrightarrow f^n(0) \nrightarrow \infty$ (where $f(z) = z^2 + c$).

Properties of the Mandelbrot Set

1. M is bounded, in fact $M \subset B(0,3)$.
2. M has a proper interior, i.e. $B(0,\frac{1}{4}) \subset M$.
3. M is closed, and then is compact.
4. M is connected.
5. M might be locally connected. (This is unsolved)
6. Hausdorff dimension of the boundary of M equals 2.

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