Notes on Aperiodic Order

whzecomjm

Jul 30, 2015

Contents

1 Introduction to Aperiodic Order

The mathematics of aperiodic order is a very young mathematical discipline, which arose from the discovery of quasicrystals in the early 1980's. The physical motivation is the introduction and study of mathematical models of quasicrystals. Independently of the relevance to physics and more generally, one is interested in features of order in the absence of periodicity.

Danny Schechtman, from Technion, observed in 1982 a strange metallic material (alloy of Aluminium and Manganese): its diffraction pattern looked like that of a crystal (with sharp, bright spots), but showed a "forbidden" 10-fold rotational symmetry. "Forbidden", because crystals were always assumed to be periodic arrangements of atoms, and such an arrangement can only have rotational symmetry of order 2, 3, 4, or 6. The revolutionary insight of Schechtman was that he didn't dismiss his result as "experimental error", but realized that this was indeed a new kind of material, which he called "aperiodic crystal", and now more commonly called "quasicrystal". For his discovery he won the Nobel Prize in Chemistry in 2011 [5].

Many physicists and mathematicians started to work on developing adequate models for quasicrystals. Interestingly, one of the most relevant models had been already discovered earlier by Roger Penrose motivated by problems in Logic. Penrose tilings have been publicized by Martin Gardner in 1977 in the popular magazine Scientific American. These tilings have many remarkable features, and among them the fact that their diffraction pattern (appropriately defined) looks very much like the one of a quasicrystal, with 10-fold rotational symmetry.

In this notes, we will explore some of the mathematical aspects of this theory based on lecture notes given by B. Solomyak. We start with a few historical remarks, mostly taken from the book chapter by E. A. Robinson, Jr [4].

Hao Wang (1961) considered following problem for squares with colored edges, which became known as "Wang tilings."

> **Tilling Program**: Is there an algorithm that, upon being given a set of prototiles, with matching rules, decides whether a tiling of the entire space exists?

When $d = 1$, the "Wang tiles" are just intervals with colored endpoints, and there is an easy algorithm to answer the Tiling Problem. Draw a graph whose vertices are prototiles and directed edges indicate which pairs are allowed. A tiling of $\mathbb{R}$ exists if and only if there is an infinite path in this graph, which is equivalent to existence of a cycle.

A tiling T of $\mathbb{R}^d$ is called a periodic tiling if its translation group $\Gamma_T = \{t \in \mathbb{R}^d : T - t = T\}$ is a lattice (free abelian group), that is, a subgroup of R^d with d linearly independent generators. A tiling is called aperiodic if $\Gamma_T = \{0\}$.

From the discussion above it follows that if a tiling of $\mathbb{R}$ with a given prototile set exists, then there is a periodic tiling. Wang conjectured that the same holds for $d > 1$. More precisely, he conjectured that (1) there is an algorithm that decides the Tiling Problem; (2) if a tiling exists, then there exists a periodic tiling. Wang proved that (2) implies (1). However, the conjecture turned out to be false! Wang's student, Robert Berger (1966) proved that the Tiling Problem is undecidable and constructed an "aperiodic tiling system," that is, a prototile set which tile the plane but only aperiodically.

One of the most interesting aperiodic sets is the set of Penrose tiles, discovered by Roger Penrose [5]. Penrose tilings play a central role in the theory because they can be generated by any of the three main methods: **local matching rules, tiling substitutions, and the projection method**. To get a good intuition of the Penrose tilings, one could find some pictures of two generators in Wikipedia.

1.1 Sturmian Sequences

We will consider sequences in a finite alphabet $\mathcal{A}$. Denote by $\mathcal{A}_n$ the set of "words" of length n in the alphabet $\mathcal{A}$. Given an infinite sequence $u \in \mathcal{A}^{\mathbb{N}}$, let $L_n(u)$ be the set of words of length n which occur in u, which is called language of u. The cardinality $p_u(n) = \#L_n(u)$ is called the complexity of a sequence u.

Here are some propositions of the complexity of a sequence for periodic property etc.

1. If u is an eventually periodic sequence, then $p_u(n)$ is bounded.
2. If there exists n such that $p_u(n) \leq n$, then u is eventually periodic (in which case $p_u(n) \leq C$ for a constant C).

Thus, $p_u(n) = n + 1$ is the minimal possible complexity of a non-periodic sequence.

A sequence $u \in \{0, 1\}^{\mathbb{N}}$ is called Sturmian if $p_u(n) = n + 1$ for all n.

Every Sturmian sequence u is recurrent, which means that every word that occurs in u appears infinitely often.

An easy example is the Fibonacci sequence obtained from the substitution $\zeta : 0 \to 01, 1 \to 0$ by $u = \lim_{n\to\infty} \zeta^n(0) = 01001010010\ldots$

Another Sturmian sequence is so called rotation sequence, let u be an infinite sequence of 0s and 1s. The sequence u is Sturmian if for some $beta \in [0, 1)$ and some irrational $\alpha \in (0, 1)$. Then

$$u_n = \lfloor n\alpha + \beta \rfloor - \lfloor (n-1)\alpha + \beta \rfloor$$

for all n or

$$u_n = \lceil n\alpha + \beta \rceil - \lceil (n-1)\alpha + \beta \rceil$$

for all n is called the rotation sequence with rotation angle α and initial point β.

For example, the Fibonacci sequence is the rotation sequence with $\alpha = 1-\varphi^{-1} = 0.381\ldots$ and $\beta = \varphi^{-2}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. In fact, every Sturmian sequence is also rotation sequence.

1.2 Substitutions

Let A be a finite alphabet of cardinality $m \geq 2$, usually $\mathcal{A} = \{0, \ldots, m-1\}$ or $A = \{1, \ldots, m\}$. Denote by $\mathcal{A}^+$ the set of all nonempty "words" using the "letters" from $\mathcal{A}$, and let $\mathcal{A}^* = \mathcal{A}^+ \cup \{\epsilon\}$, where ϵ is the empty word. We will also write $\mathcal{A}^{\mathbb{N}}$ to denote the set of (one-sided) sequences: $u = u_0u_1u_2\cdots \in \mathcal{A}^{\mathbb{N}}$ whenever $u_j \in \mathcal{A}$ [6]. (Our convention is that $0 \in \mathbb{N}$.)

A substitution on $\mathcal{A}$ is a map $\zeta : \mathcal{A} \to \mathcal{A}^+$. It extends to a map on $\mathcal{A}$ and $\mathcal{A}^{\mathbb{N}}$ by concatenation. A fixed point for ζ is a sequence $u \in \mathcal{A}^{\mathbb{N}}$ such that $\zeta(u) = u$. A periodic point is u such that $\zeta^k(u) = u$ for some k.

Note that this is periodicity with respect to the substitution, and not the usual periodicity of a sequence.

For a substitution ζ on an alphabet of m symbols the substitution matrix is defined by

$$S_\zeta(i,j) = \text{number of i's in } \zeta(j).$$

In our example of Fibonacci sequence above we have

$$S_\zeta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The canonical homomorphism is the map $l : \mathcal{A}^* \to \mathbb{N}^m$ defined by $l(W) = (l_i(W))_{i=1}^m$, where $l_i(W)$ is the number of symbols i is the word W.

We consider $l(W)$ as a column vector. Note that we can express the substitution matrix in terms of its columns as follows:

$$S_\zeta = [l(\zeta(1)), l(\zeta(2)), \ldots, l(\zeta(m))]$$

We obtain the following important identities:

$$l(S_\zeta W) = S_\zeta(l(W)), S_{\zeta^k} = S_\zeta^k.$$

The transformation $W \mapsto S_\zeta W$ is often called the Abelianization of the substitution.

A substitution ζ is called primitive if there exists $k \in \mathbb{N}$ such that for all $a \in \mathcal{A}$, the word $\zeta^k(a)$ contains all symbols $b \in \mathcal{A}$. In view of the above, this is equivalent to the condition that S_ζ^k has all entries strictly positive. (Such matrices are also called primitive.)

1.3 Perron-Frobenius Theorem

Theorem (Frobenius-Perron). Let M be a primitive $d \times d$ matrix. Then

1. there exists a dominant eigenvalue $\theta > 0$, which is simple and strictly greater than any other eigenvalue in absolute value;
2. the eigenvector corresponding to θ is strictly positive (component-wise): $\vec{x} > \vec{0}$;
3. for every non-negative vector $\vec{y} \geq \vec{0}$, the sequence of vectors $M^n\vec{y}$ approaches $\vec{x}$ in direction, more precisely,

$$\lim_{n\to\infty} \frac{M^n y}{\|M^n y\|} \to \frac{x}{\|x\|}.$$

One could find a proof in [5].

An algebraic integer θ whose Galois conjugates, i.e. the other roots of the minimal polynomial, are strictly less than θ in absolute value, is called a Perron number.

Thus, Perron-Frobenius Theorem implies that the dominant eigenvalue of a primitive integer matrix is a Perron number. It is interesting that the converse is also true! This is important in symbolic dynamics.

Theorem (D. Lind, 1986). For any Perron number θ there exists a primitive matrix M such that θ is the PF eigenvalue of M.

1.4 Letter and Word Frequencies

Let W be a word, $|W| = k$, and u a sequence in a finite alphabet $\mathcal{A} = \{1, \ldots, m\}$. The frequency of W in u is defined by

$$\mathrm{freq}_u(W) = \lim_{N\to\infty} \frac{\#\{i \leq N-k : u[i, i+k-1] = W\}}{N},$$

assuming the limit exists.

We say that the frequency of W in u exists uniformly if

$$\mathrm{freq}_u(W) = \lim_{N\to\infty} \frac{\#\{i \in [l, l+N-k-1] : u[i, i+k-1] = W\}}{N},$$

uniformly in $l \in \mathbb{N}$.

Let u be a fixed point of a primitive substitution ζ on the alphabet $\mathcal{A} = \{1, 2, \ldots, m\}$. Then

1. for any $W \in L(u)$, the frequency $\mathrm{freq}_u(W)$ exists uniformly;
2. Let $x_i = \mathrm{freq}_u(i)$ (that is, the frequency of a single letter i). Then $\vec{x}$ is the PF eigenvector of S_ζ, normalized.

There is an easy criterion for nonperiodicity.

Theorem (Pansiot). Let ζ be a substitution of constant length, which is 1-to-1 on the alphabet $\mathcal{A}$. Let u be a fixed point of ζ. It is non-periodic if and only if there is a letter $\alpha \in \mathcal{A}$ which appears in u with at least two distinct right extensions $\alpha\beta, \alpha\gamma$ (note that one of β, γ may equal α, we just require that $\beta \neq \gamma$).

2 Symbolic dynamical systems

2.1 Symbolic dynamical systems

Let's start with a topological dynamical system, which is a topological space, together with a continuous transformation, a continuous flow, or more generally, a semigroup of continuous transformations of that space. It is called invertible if T is a homeomorphism. A symbolic dynamical system which we will discuss later is somehow a special topological dynamical system. For simplicity, we assume that X is a compact metric space.

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^{\mathbb{N}}$ be the space of the sequences as an infinite products of finite sets with discrete topology with a metric

$$d(x, y) = 2^{-\min\{n \in \mathbb{N} : x_n \neq y_n\}}.$$

In other words, two sequences x and y are within a distance of at most 2^k if and only if they coincide up to index k (or coincide from -k to +k in the 2-sided case). $\mathcal{A}^{\mathbb{N}}$ is a compact metric space, which is topologically a Cantor set (compact, totally disconnected, without isolated points). Then, we consider the one-sided left shift map T defined by $(Tx)_n = x_{n+1}$, for all $n \in \mathbb{N}$.

Note that T is continuous, it is invertible on $\mathcal{A}^{\mathbb{Z}}$ and surjective, but non-injective on $\mathcal{A}^{\mathbb{N}}$. A **symbolic dynamical system** is a topological space (X, T), such that X is a closed subset of $\mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$ which is a T-invariant, with T the left shift. A symbolic dynamical system associated with u is the orbit closure of u which is a compact T-invariant.

A sequence u is said to be **uniformly recurrent (or minimal)** if every word that occurs in u occurs with bounded gaps. Equivalently, for every n there exists

K, such that $u_0, \ldots, u_n$ occurs in every subword of u of length K. A topological dynamical system is called minimal if every orbit is dense, or equivalently, there are no nontrivial closed invariant subsets.

The following theorem for the relationship between uniform recurrence and minimality is well-known:

Theorem. A sequence u is uniformly recurrent if and only the associated symbolic dynamical system is minimal.

To prove those theorem, we need the following lemma.

Lemma Let $u \in \mathcal{A}^{\mathbb{N}}$. Consider $X_u = \text{clos}\{T^n u : n \geq 0\}$, the orbit clouser of u. Then $x \in X_u$ if and only if $\mathcal{L}(x) \subset \mathcal{L}(u)$, where $\mathcal{L}(x)$ denotes the language of x (the set of all finite words which appear in x).

In view of the last lemma, (X_u, T) is minimal whenever $\mathcal{L}(x) \subset \mathcal{L}(u)$ implies $\mathcal{L}(x) = \mathcal{L}(u)$.

Corolary. Let u be a fixed point of a primitive substitution. Then u is uniformly recurrent.

2.2 Topological dynamical system

We continue the topics on topological dynamical systems and introduce a very important theorem. Let $\mathcal{M}_X$ be the set of Borel probability measures on X. A measure $\mu \in \mathcal{M}_X$ is said to be invariant for T, or T-invariant if

$$\mu(T^{-1}B) = \mu(B) \text{ for all Borel sets } B \subset X.$$

Theorem(Krylov and Bogoliubov). For every topological dynamical system there is at least one invariant measure.

Definition. A topological dynamical system (X, T) is called uniquely ergodic (UE) if it has **only one** invariant probability measure.

Example. Irrational circle rotation is minimal and uniquely ergodic. More generally, consider a translation on the d-torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$.

Theorem(Kronecker). The toral translation T_α is minimal(uniformly recurrent) if and only if $\{1, \alpha_1, \ldots, \alpha_d\}$ are linearly independent over the rationals.

Theorem(Weyl). The toral translation T_α is minimal if and only if it is uniquely ergodic. (The unique invariant probability measure is the Lebesgue = Haar measure on the torus.)

2.3 Measure-preserving dynamical systems

Definition. Let $(X, \mathcal{B}, \mu)$ be a probability measure space. A measurable transformation $T : X \to X$ is measure-preserving if $\mu(T^{-1}E) = \mu(E)$ for all $E \in \mathcal{B}$.

We will abbreviate by writing that is a m.-p. s and sometimes call it as measure-theoretic dynamical systems. It is called invertible if T^{-1} exists μ-a.e. and is measurable. An m.-p. s. $(X, \mathcal{B}, \mu, T)$ is ergodic if $T^{-1}B = B, B \in \mathcal{B}$, implies $\mu(B) \in \{0, 1\}$.

Lemma. (1) A system $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if every T-invariant measurable function f is constant.
(2) A uniquely ergodic system is ergodic with respect to the unique invariant measure.

Theorem.(Von Neumann's Mean Ergodic Theorem). For a m.-p. s. $(X, \mathcal{B}, \mu, T)$, we have

$$\frac{1}{N} S_N f(x) := \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n \to Pf, \text{ as } T \to \infty, \text{ for all } f \in L^2(X, \mu),$$

where P is the orthogonal projection onto the subspace of T-invariant L^2-functions, and the convergence is in L^2.

Theorem.(Birkhoff's Pointwise Ergodic Theorem). For an ergodic m.-p. s. $(X, \mathcal{B}, \mu, T)$, we have

$$\frac{1}{N} S_N f(x) \to \int_X f d\mu \text{ as } T \to \infty, \text{ for all } f \in L^1(X, \mu),$$

where convergence is almost everywhere and in L^1.

Theorem. Let $T : X \to X$ be a continuous map on a compact metric s-pace. Then T is uniquely ergodic if and only if for every $f \in C(X)$, we have $\frac{1}{N} S_N f(x) \to C_f$, where the constant C_f is independent of x. Moreover, $C_f = \int_X f d\mu$, where μ is the unique invariant probability measure, and the convergence is uniform.

2.4 Elements of spectral theory

Let $(X, \mathcal{B}, \mu, T)$ be a m.-p. s. The operator $U_T : f \mapsto f \circ T$ on $L^2(X, \mu)$ is called the **Koopman operator** associated with T. The Koopman operator is an isometry. If T is invertible, then U_T is unitary (i.e. U_T{-1}=U_T*) and $U_T^{-1} = U_{T^{-1}}$. Spectral theory of the measure-preserving transformation is the spectral theory of the Koopman operator U_T.

For spectral theory, we also have eigenvalue like in linear algebra. A complex number λ is an eigenvalue of U_T if there exists $f \in L^3(X, \mu)$, call eigenfunction, such that $U_T f = \lambda f$. Equivalently, $f(Tx) = \lambda f(x)$ a.e. Note that $\lambda = 1$ is always an eigenvalue, corresponding to the "trivial" constant eigenfunction.

Lemma. (1) A measure-preserving system is ergodic if and only if $\lambda = 1$ is a simple eigenvalue, i.e. with $\dim(ker(U_T - I)) = 1$.

(2) If T is ergodic, then every eigenvalue is simple, and all eigenfunctions have constant modulus.
(3) Eigenvalues of ergodic T form a group (a subgroup of the circle $\mathbb{T}$).

The spectrum of T is said to be (pure) discrete if there is a Hilbert space basis for $L^2(X, \mu)$ consisting of eigenfunctions. The spectrum of T is said to be continuous if $\lambda = 1$ is the only eigenvalue (which is simple). A good example of discrete spectrum is that The circle rotation R_α has discrete spectrum. While The doubling map T_2 has continuous spectrum.

Theorem(Halmos-Von Neumann). (1) Two invertible and ergodic m.-p. s. with identical discrete spectrum are measure-theoretically isomorphic.
(2) very m.-p. s. with discrete spectrum is measure-theoretically isomorphic to a translation on a compact Abelian group, with the Haar measure.

Definition. Let (X, T) be a topological dynamical system. A function $f \in C(X)$ is called a continuous eigenfunction, with eigenvalue λ, if $f(Tx) = \lambda f(x)$ for all x. THe system is said to be a topological discrete spectrum if the eigenfunctions span a dense subset of $C(X)$.

Example (Toral endomorphisms). Let A be an integer $d \times d$ matrix. Consider the transformation

$$T_A(x) = Ax \mod \mathbb{Z}^d, x \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d.$$

It's well-defined on the equivalence class of x. When A has non-zero determinant, T_A is a toral endomorphism, and when $\det(A) = \pm 1$, it is a toral automorphism. Note that $(T_A)^{-1} = T_{A^{-1}}$ in the latter case. Toral endomorphisms preserve the Haar measure m_d.

Theorem. Let A be an integer matrix, with determinant $|\det(A)| = 1$. Consider the measure-preserving system $(\mathbb{T}^d, T_A, m_d)$. Then
(1) the system is ergodic if and only if none of the eigenvalues of A is a root of unity;
(2) if the system is ergodic, then it has continuous spectrum.

Definition. Let μ be a Borel probability measure on $\mathbb{T}$. The Fourier coefficients $(\hat{\mu}(n))_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ are defined by

$$\hat{\mu}(n) = \int_{\mathbb{T}} \epsilon^{2\pi int} d\mu(t), n \in \mathbb{Z}.$$

A sequence $(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is *positive definite* if for any complex sequence $(z_j)_{j\geq 1}$, we have $\forall n \geq 1$,

$$\sum_{1\leq i,j\leq n} z_i \bar{z}_j a_{i-j} \geq 0.$$

Theorem (Bochner-Herglotz). Any positive definite sequence is the sequence of Fourier coefficients of a positive finite Borel measure.

3 Tiling Dynamical System and Delone Sets

3.1 Tiling Dynamical System

We consider tilings of the Euclidean space $\mathbb{R}^d$. Tile is a compact set, which is a closure of its interior. More precisely, it is a pair (A, i), where A is a compact set and i is its label. This is needed in order to distinguish between geometrically identical tiles. A **prototile set** id a finite set of tiles $\mathcal{A} = \{T_1, \dots, T_m\}$; this is an analogue to "alphabet" in dynamical system. A **Patch** is a finite set of tiles which have disjoint interiors. Denote by $\mathcal{P}_{\mathcal{A}}$ the set of patches whose every tile is a translate of one of the prototiles. *Tiling* is a set of tiles with disjoint interiors, whose union is $\mathbb{R}^d$. We usually assume that all tiles are translates of one of the prototiles.

Definition (FLC). A tiling $\mathcal{T}$ has **finite local complexity** (FLC) if there are finitely many $\mathcal{T}$-patches (its "size" can be measured by the diameter of its support) of any given "size" up to translation.

Definition (repetitive). A tiling is repetitive (uniform recurrence) if for every patch P there exists $R = R(P)$ such that there is a translate of P in every ball of radius R.

Now we will define a tiling dynamical system. Firstly, we should determine the metric. Tiling metric is defined as follows: two tilings are ϵ-close, for a small $\epsilon > 0$ whenever they agree on a ball $B_{1/\epsilon}(0)$, up to a translation of size $\leq \epsilon$. Denote the distance between $\mathcal{T}_1$ and $\mathcal{T}_2$ by $d(\mathcal{T}_1, \mathcal{T}_2)$.

Definition (Tiling Space). A tiling space is a closed, translation invariant set of tilings. One can consider tiling spaces $X_{\mathcal{T}}$ as the set of all tilings with tiles that are translates of the prototiles in $\mathcal{T}$. Symbolically,

$$X_{\mathcal{T}} := \overline{\{\mathcal{T} - g : g \in \mathbb{R}^d\}}.$$

Theorem. The tiling metric is complete. If $\mathcal{T}$ has FLC, then the tiling space $X_{\mathcal{T}}$ is compact and vice verse.

Definition (Tiling dynamical system). The group $\mathbb{R}^d$ acts on $X_{\mathcal{T}}$ by translations $T_g : \mathcal{S} \mapsto \mathcal{S} - g$. This is a continuous action, and we call the resulting system $(X_{\mathcal{T}}, T_g)_{g \in \mathbb{R}^d}$ the (topological) tiling dynamical system associated with $\mathcal{T}$. We usually simply write $(X_{\mathcal{T}}, \mathbb{R}^d)$. A tiling $\mathcal{T}$ is so-called **aperiodic** if $\mathcal{T} - g = \mathcal{T}$ only for $g = \vec{0}$.

Theorem. The tiling dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is minimal iff $\mathcal{T}$ is repetitive.

3.2 Delone Sets and Associated Dynamical Systems

Definition (Delone sets) A Delone set $\Lambda \subset \mathbb{R}^d$ is a uniformly discrete, relative dense set. More precisely, there exists $0 < r < R < \infty$ such that every ball of radius r contains at most one point of Λ and every ball of radius R contains at least one point of Λ.

We can define a metric on the space of Delone sets, similarly to tilings, and consider resulting spaces and dynamical systems. Instead of "patches" for tilings we will talk about "patterns" for Delone sets (i.e., finite subsets). Sometimes we also consider "colored" Delone sets, or more precisely, Delone multisets. This is more in line with tiling theory.

There are three important properties of Delone sets, which are relevant for us:

- $\Lambda - \Lambda$ is discrete, which means every point is isolated, iff there are finitely many points in every ball. In fact, this is equivalent to finite local complexity (finitely many patterns of diameter $< R$, for any R, up to translation), because two-point patterns determine all patterns.
- $[\Lambda] := \mathbb{Z}[x : x \in \Lambda]$, the abelian group (subgroup of $\mathbb{R}^d$) generated by Λ is finitely generated.
- $\Lambda - \Lambda$ is uniformly discrete iff $\Lambda - \Lambda$ is Delone. Such Λ are called *Meyer sets*.

Proposition. If $\Lambda - \Lambda$ is discrete, then $[\Lambda]$ is finitely generated, but the converse if false.

3.3 Curtis-Hedlund-Lyndon theorem

The Curtis-Hedlund-Lyndon theorem is a fundamental theorem in symbolic dynamical system, which is a mathematical characterization of (one-dimensional) cellular automata in terms of their symbolic dynamics. A generalization to higher dimensional integer lattices was soon afterwards published by Richardson (1972), and it can be even further generalized from lattices to discrete groups.

We first give some definitions. Let $\mathcal{A} = \{0, \dots, m-1\}$ be a finite alphabet. Recall that a (2-sided) symbolic dynamical system over $\mathcal{A}$ is (X, T), where $X \subset A^{\mathbb{Z}}$ is closed and T-invariant, T being the left shift map. For more details about symbolic dynamical system, please refer to the post: symbolic dynamical system.

Suppose $(X, T), (Y, S)$ are two symbolic dynamical systems, a map $F : X \to Y$ is called a **code** if it is continuous and intertwines the shifts, i.e. $F \circ T = S \circ F$. We think of a code as a homomorphism of two dynamical system. As analogies of homomorphism, if a code F is surjective, then it's called a quotient map or factor map or epimorphism of dynamical systems; if it is injective, then it is called an embedding or monomorphism; if it is bijective, then it is an isomorphism or conjugacy.

Let $\phi : \mathcal{A}^{2n+1} \to \mathcal{A}$, the **sliding block code** corresponding to ϕ is a map $F_\phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by

$$y = (y_k)_{k \in \mathbb{Z}} := F_\phi(x), \quad y_k = \phi(x_{k-n}, \dots, x_{k+n}).$$

It is easy to see that the map F_ϕ produced in this way will be continuous and commute with the shift. That is, such a map F_ϕ defined a code.

The Curtis-Hedlund-Lyndon theorem asserts that every code is a sliding block code in the above language.

Theorem (Curtis-Hedlund-Lyndon). Let (X, T) and (Y, S) be two symbolic dynamical systems over $\mathcal{A}$, and suppose that $F : X \to Y$ is a continuous map commuting the shifts, namely F is a code, the F is a sliding block code.
Proof. By continuity of F, for each symbol a in the alphabet of S, the inverse image of the closed open set $\{x : x_0 = a\}$ is closed open, i.e. compact open. Any open set is a union of cylinders and any compact open set is a union of finitely many cylinders. If a cylinder C is defined on coordinates $[j, k]$ and $j' < j < k < k'$, then C is a union cylinders defined on coordinates $[j', k']$. Because the alphabet $\mathcal{A}$ is finite, it follows that we may choose N sufficiently large that $\forall a$, the inverse image of $\{x : x_0 = a\}$ is a union of cylinders defined on coordinates $[-N, N]$.

Now we define $\phi(x_{-N} \dots x_N) = a$ if $\{x : x_{-N} \dots x_N\}$ is contained in the inverse image of $\{x : x_0 = a\}$. Because F commutes with the shift, we have for any i such that

$$\begin{aligned}(F(x))_i &= (S^{-i} F T^i x)_i = (F T^i x)_0 \\ &= \phi([T^i x]_{-N} \dots [T^i x]_N) \\ &= \phi(x_{i-N} \dots x_{i+N}). \square\end{aligned}$$

References

[1] Blanchard F., Maass A., Nogueira A. Topics in Symbolic Dynamics and Applications. pp57–61.
[2] B. Solomyak, Tilings and Dynamics, available at https://www.math.washington.edu/~solomyak/PREPRINTS/notes6.pdf
[3] B. Solomyak, Mathematics of Aperiodic Order,2015.
[4] E. A. Robinson, Jr., Symbolic dynamics and tilings of $\mathbb{R}^d$, in Symbolic dynamics and its applications, Proc. Sympos. Appl. Math., Vol. 60, Amer. Math. Soc., Providence, RI, 2004, pp. 81–119.
[5] R. Penrose, The role of aesthetics in pure and applied mathematical research, Bull. Inst. Math. Appl.10 (1974), 266-271. Pentaplexity: a class of nonperiodic tilings of the plane, Math. Intelligencer 2(1979/80), no. 1, 32–37.
[6] M. Boyle, Notes on the Perron-Frobenius Theory of Nonnegative Matrices.